Seminar on homological stability for linear and diffeomorphism groups

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Introduction

Suppose that $G_1 \to G_2 \to \dots$ is a sequence of groups and homomorphisms. We say that the sequence satisfies homological stability if the induced maps $H_q(BG_n) \to H_q(BG_{n+1})$ are isomorphisms for $q \leq f(n)$, where $f(n)$ tends to ∞ with increasing n. In practise, the condition will be $q \leq n/a + b$ with $a, b \in \mathbb{N}$. Slightly more generally, we can ask the same for a sequence X_n of spaces.

Such theorems have been established for an astonishing variety of groups or spaces, including symmetric groups, linear groups, orthogonal groups, diffeomorphism groups of highly-connected high-dimensional manifolds and mapping class groups of oriented surfaces, to name only these results that we will discuss. Besides these, there are results for automorphism groups of free groups ([HW05], [HV04]), mapping class groups of 3-manifolds ([HW10]), mapping class groups of non-orientable surfaces [Wah08] and configuration spaces of manifolds [RW13]. For many of these groups, there are several different extended version to nonconstant coefficients and various decorations. In many of the above cases, the stable homology is known ([Nak61], [Qui72], [Bor74], [Gal11], [GRW14], [MW07], [GTMW09]), but we leave these questions aside as they require entirely different methods.

All the proofs follow a blueprint that was introduced by Quillen in the context of linear groups [Qui72]. The crucial point is to find a highly connected simplicial complex C_n with an action of G_n, such that the action of G_n on the p-simplices is transitive and the stabilizers are smaller groups G_{n-p-1}. A careful study of the spectral sequence of the action proves stability. We discuss four sequences of groups (or spaces) in this seminar, in increasing degree of complexity: symmetric groups, general linear groups, orthogonal groups, diffeomorphism groups of highly connected high-dimensional manifolds, mapping class groups of surfaces.

As a warm-up, we discuss symmetric groups, thereby introducing the spectral sequence argument (talk 1). The next talk 2 introduces the class of simplicial complexes we will mainly talk about. The following talks, 3 and 4 establish the connectivity results for the complexes used for general linear and orthogonal groups (among these complexes is the Tits building of a vector space). The methods are quite different from each other and the third one depends on both of the previous ones. The following talk 5 harvests the results and proved stability for linear and orthogonal groups. So far, we saw mainly complexes and groups of an algebraic nature, but from talk 6 on, we move to diffeomorphism groups. The first case is the recent result by Galatius and Randal-Williams on diffeomorphisms of highly-connected high-dimensional (dim ≥ 6) manifolds. This makes crucial use of the connectivity result of talk 4. The last part of the seminar is devoted to the more classical case of mapping class groups. This is originally due to Harer [Har85] and

*The seminar program builds on a program for a similar seminar run in Münster by Prof. Johannes Ebert and Prof. Michael Weiss. We thank them for letting us use their program.

was later improved by Ivanov [Iva89, Iva93], Boldsen [Bol12] and Randal-Williams [RW16] and others. Here we follow the exposition [Wah13].

Organization

The seminar will meet on Thursdays, 14:15 - 16:00, seminar room 1.007 at the math department.

1 Techniques

Talk 1. (Warm-up: homological stability for symmetric groups), David Bowman, April 11

Homological stability for symmetric groups was established by Nakaoka [Nak61] as a corollary of his actual computation of the homology. A different proof was given by Segal [Seg79]. There are other proofs that follow the blueprint and this talk should discuss these proofs. There is an algebraic version [Ker05] and a geometric version [RW13], Section 5 . The arguments are really isomorphic. Nevertheless, present both versions to a certain extent, as this sheds light on the machinery. The key is the high connectivity of the semisimplicial set denoted $C_*(m)$ in [Ker05] or $F(C)$ in [RW13]. The focus of this talk should lie on the way the spectral sequence is built and used, so if there is a time problem, sacrifice the proof of the connectivity theorem.

Talk 2. (Cohen-Macaulay-complexes), Mathis Birken, April 18

Most of the simplicial complexes we will see are Cohen-Macaulay complexes and this talk should introduce them. Begin with a general discussion of simplicial complexes versus Δ-sets versus partially ordered sets, as all three notions and their relationship is used later on. Then introduce the Cohen-Macaulay-condition. Source: [Qui78], Sections 1,8,9. Then present the 'poset fibre theorem'. The version needed in talk 5 is Proposition 1.2 of [Cha87], a slight strengthening of Theorem 9.1 in [Qui78]. The talk should conclude by introducing the main characters of the following talks: the posets (complexes) of [vdK80], Thm. 2.6, the Tits building of a vector space [Qui78], page 118, the complex of Thm 1.6 of [Vog81], [Cha87], Theorems 2.9 and 3.2. State the connectivity result in each case.

2 Linear and orthogonal groups

Talk 3. (Van der Kallen-Maazen's connectivity theorem), Nick Nordwald, April 25

This talk should discuss the proof of Theorem 2.6 of [vdK80] in full detail. This is used to establish stability for general linear groups and also to deduce Charneys theorem in the next two talks. Please discuss the ring- and module theoretic basics as well, but keep in mind that we do not loose much if the ring is restricted to be a field or principal ideal domain. Some of the technical results of loc. cit. have been improved by [Cha84], Section 1 (quoted as Lemmata 1.4, 1.5, 1.6 in [Cha87]). As they are used later in talk 5 and as Lemma 1.5 implies Theorem 1 of [Ker05] that has been used in talk 1, you should try to combine both sources [vdK80] and [Cha84]. The original source is Maazen's unpublished thesis.

Talk 4. (Charney's connectivity theorem), Elena Ertle, May 2

Charney's connectivity theorem ([Cha87], Theorem 3.2) will be used for stability of orthogonal groups and also for diffeomorphism groups of high-dimensional manifolds. Discuss the more general and simpler proof given in [GRW18], Theorem 3.2. Introduce the necessary background on quadratic forms on modules over general rings.

Talk 5. (Spectral sequence argument for linear and orthogonal groups), Fabio Neugebauer, May 16

The complexes studied in the previous talks are used to prove stability for linear and orthogonal groups, [Cha84], Section 4, and [vdK80], Sections 3,4,5. Discuss these results using the axiomatic framework of [RWW17] and Charney's connectivity theorem proved in the previous talk.

3 Diffeomorphism groups of highly connected high-dimensional manifolds

This part is based on the preprint [GRW12] which was later generalized to the paper [GRW18]. For simplicity we suggest that you focus on the connected sums of $S^n \times S^n$ [GRW12] (and perhaps only mention the more general theorem). Note that we already used parts of [GRW18] for the proof of Charney's theorem in Talk 4.

Talk 6. (Statement of the result and spectral sequence argument), Brais Gerpe Vilas, June 6

State the result [GRW12], Theorem 1.2. Then define the complex that is used (Definition 4.1) and state the connectivity result (Theorem 4.6) which will be proven in the next talk. Then present the spectral sequence argument (which is rather straightforward in this case), Section 5. You also should give some background on the classification of highly connected high-dimensional manifolds.

Talk 7. (Connectivity result of Galatius–Randal-Williams), Congzheng Liu, June 13

Proof of the connectivity result Theorem 4.6 of [GRW12]. The proof is in Sections 1,2,3,4 (most of Section 3 has been covered in talk 4). The proof works from Charney's algebraic result (for which we used the proof from [GRW18] in talk 4) to a complex of embedded spheres, via the Whitney trick and the simplicial complex techniques of Section 2.1, to prove connectivity of a discretized version. To get at the version of interest (Theorem 4.6) one uses microfibrations and a new technique. Even if you like to avoid the technicalities, describe the method precisely. The same method can be used in Talk 12.

4 Mapping class groups

Talk 8. (Theorems of Earle-Eells and Gramain), Hoi Pak Lau, June 20

Give the definition of mapping class groups of surfaces, as well as a short survey that includes Dehn twists and the classification of smooth surfaces; this is an essential ingredient for the stability theorem. The rest of the proof is devoted to another fundamental ingredient: the theorem by Earle-Eells [EE69] that asserts that the components of the diffeomorphism group of a surface are contractible. Give a short description of how this fits into Teichmüller theory; a full proof belongs to the realm of geometric analysis and is well beyond the scope of this seminar. There is a topological proof of this fact by Gramain [Gra73]. Show how the Earle-Eells theorem is equivalent to Theoreme 5 of [Gra73]. If time permits, say something on the topological proof of that fact.

Talk 9. (Statement of the Harer stability theorem), Jan Sneeuw, June 27

Statement of the stability theorem, see [Wah13], Theorems 1.1 and 1.2. Introduce the simplicial complexes that are used for the proof (Definition 2.1) and state the connectivity result Proposition 2.8, whose proof covers the next two talks. Then go through the proofs of the other ingredients in Section 2 in detail.

Talk 10. (Connectivity argument I), Mathieu Wydra, July 4

Explain the different steps of the connectivity argument (first page of Section 4, [Wah13]). Then prove Theorem 4.1 of [Wah13]. See also [HV17] for an axiomatic treatment. You might want to consult the original source [Hat91].

Talk 11. (Connectivity argument II), Maximilian Hans, July 11

Theorems 4.3, 4.8 and 4.9 of [Wah13] (the last one is what is needed for the stability theorem). See also [HV17] for an alternative source.

Talk 12. (Spectral sequence argument), Julius Groenjes, July 18

[Wah13], Sections 3 and Sections 5. The spectral sequence argument for Theorem 1.1 is a bit more sophisticated than that of Talks 1, 5 and 6 but should pose no problems by now. Finally, give the proof of Theorem 1.2. This proof is due to [RW16], Section 11, and it can be simplified using the technique of [GRW18], Theorem 4.6. You might want to read the original proof, e.g., [Har85]: it is much more involved.

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